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ON AVERAGING IN SYSTEMS WITH A VARIABLE NUMBER OF DEGREES OF FREEDOM*

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An averaging procedure is established for systems with a variable number of degrees of freedom which arise when considering vibrocollisional oscillations with zero velocity restitution coefficient. Compared with the method of staged integration /1, 2/ the approach presented, associated with non-analytical changes in the variables /3, 4/, widens the class of systems that can be considered. Unlike the classical averaging method /5-7/ there is a reduction in the degenerate degrees of freedom because of the specific degeneracy of the problem.

1. Consider a system described over certain times by differential relations, and at other times interval by differential and finite relations of the following form:

$$x^{\cdot} = \mu X(x, yM, t, \mu), \quad y^{\cdot}M = \mu Y(x, yM, t, \mu)\, M$$
$$y(2\pi n) = G(x(2\pi n), \mu), \quad x(0) = x_0 \tag{1.1}$$

Here $M = M(t)$ is a 2π-periodic piecewise-constant function (see Fig.1); here and throughout $n = 0, 1, 2, \ldots$.

We take X and Y to be bounded 2π-periodic finite-dimensional vector functions satisfying Lipschitz conditions on their first and second arguments, G is a bounded vector function with bounded partial derivatives with respect to the first argument and μ is a small parameter.

The vector function $y(t)$ is a solution of an infinite sequence of systems of differential equations, each of which acts in the time interval $2\pi n \leqslant t < 2\pi(n+1)$, after which new initial conditions are imposed.

Together with system (1.1) we consider the averaged equations

$$\xi^{\cdot} = \mu \Xi(\xi, \eta, \mu) \equiv \mu \langle X(\xi, \eta M, t, \mu)\rangle$$
$$\eta = G(\xi, \mu), \quad \xi(0) = x_0 \tag{1.2}$$

Under the given conditions we formulate the following theorem.

Theorem. If the solution of system (1.2) is given in a time interval of the order of $1/\mu$, then

$$\|x - \xi\| \leqslant C_1\mu, \quad \|yM - \eta M\| \leqslant C_2\mu \tag{1.3}$$

during that time interval, and the constants C_1 and C_2 remain bounded as $\mu \to 0$.

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Proof. We redefine system (1.1) in the domain where $M = 0$:

$$\begin{gathered} x_1^{\cdot} = \mu X\,(x_1,\, y_1 M,\, t,\, \mu) \\ y_1^{\cdot} = \mu Y\,(x_1,\, y_1 M,\, t,\, \mu) + 2\mu H\,(x_1,\, y_1,\, \mu)\,(1 - M) \\ y_1\,(2\pi n) = G\,(x\,(2\pi n),\, \mu), \quad x_1\,(0) = x_0 \end{gathered} \tag{1.4}$$

Here $H\,(x_1, y_1, \mu)$ is an arbitrary bounded function satisfying the Lipschitz condition. It will be specified later. Because of the uniqueness of the solution of the Cauchy problem $x_1 \equiv x$ and $y_1 M \equiv yM$.

We will apply the averaging method procedure to system (1.4). We perform a pointwise change of variables

$$x_1 = \xi_* + \mu u, \quad y_1 = \eta_* + \mu v \tag{1.5}$$

where u and v are periodic functions of ξ_*, η_*, t and μ whose averages vanish. We will assume that the functions X, Y and H satisfy Lipschitz conditions and choose functions u and v as follows:

$$\begin{gathered} u = \int [X\,(\xi_*,\, \eta_* M,\, t,\, \mu) - \Xi]\, dt \\ v = \int [Y\,(\xi_*,\, \eta_* M,\, t,\, \mu) - \Psi]\, dt + 2H \int [{}^1/_2 - M]\, dt \\ \Xi = \Xi\,(\xi_*,\, \eta_*,\, \mu) \equiv \langle X\,(\xi_*,\, \eta_* M,\, t,\, \mu)\rangle \\ \Psi = \Psi\,(\xi_*,\, \eta_*,\, \mu) \equiv \langle Y\,(\xi_*,\, \eta_* M,\, t,\, \mu)\rangle \end{gathered}$$

where integration is performed with respect to the explicitly appearing time over the interval $[t_0, t]$.

We will now use the arbitrariness of the function $H\,(\xi_*, \eta_*, \mu)$ and set

$$H\,(\xi_*,\, \eta_*,\, \mu) = (\partial G\,(\xi_*,\, \mu)/\partial \xi_*)\, \Xi - \Psi$$

We then obtain

$$\begin{gathered} \xi_*^{\cdot} = \mu \Xi + \mu^2 O\,(1), \qquad \eta_*^{\cdot} = \mu\,(\partial G/\partial \xi_*)\, \Xi + \mu^2 O\,(1) \\ \eta_*\,(2\pi n) = G\,(\xi_*\,(2\pi n),\, \mu) + \mu O\,(1) \end{gathered} \tag{1.6}$$

Hence, there follow the equations

$$\eta_*^{\cdot} = G^{\cdot}\,(\xi_*,\, \mu) + \mu^2 O\,(1), \quad \eta_*\,(2\pi n) = G\,(\xi_*\,(2\pi n),\, \mu) + \mu O\,(1) \tag{1.7}$$

which are a sequence of systems of differential equations, each of which acts over a time interval $2\pi n \leqslant t < 2\pi\,(n+1)$ and has its own initial conditions. Integrating system (1.7) using the finiteness of each interval, we obtain the relation

$$\eta_* = G\,(\xi_*,\, \mu) + \mu O\,(1) \tag{1.8}$$

valid for all times.

Substituting expression (1.8) into the first equation of system (1.6) and using the fact that the function Ξ also satisfies Lipschitz conditions, we arrive at the final form of the equations

$$\begin{gathered} \xi_*^{\cdot} = \mu \Xi\,(\xi_*,\, G\,(\xi_*,\, \mu),\, \mu) + \mu^2 O\,(1) \\ \xi_*\,(0) = x_0, \quad \eta_* = G\,(\xi_*,\, \mu) + \mu O\,(1) \end{gathered} \tag{1.9}$$

System (1.9) is fully equivalent to system (1.4) over any time interval.

Considering system (1.2) alongside system (1.9), with the help of Gronwall's Lemma we obtain the required estimates (1.3) over a time interval of the order of $1/\mu$.

Remark 1°. If the functions X, Y and G are continuous at $\mu = 0$, then the system

$$\xi_0^{\cdot} = \mu \Xi\,(\xi_0,\, \eta_0,\, 0),\ \eta_0 = G\,(\xi_0,\, 0),\ \xi_0\,(0) = x_0$$

preserves the same asymptotic accuracy.

2°. The above discussion remains valid if the function M consists of a finite number of isolated impulses of arbitrary fixed length in the interval $0 \leqslant t < 2\pi$.

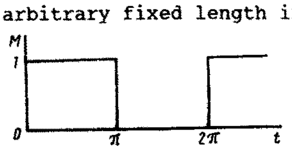

Fig.1

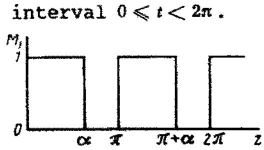

Fig.2

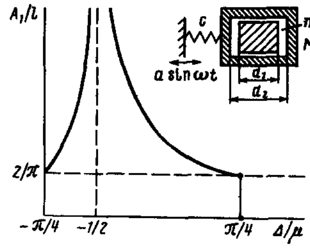

Fig.3

2. We consider a more complicated problem:

$$x^{\cdot} = \mu X, \quad y^{\cdot} M_1 = \mu Y M_1, \quad z^{\cdot} M_1 = (1 + \mu Z) M_1, \quad \varphi^{\cdot} = 1 + \mu\Phi \tag{2.1}$$
$$y(t_n) = G(x(t_n)) + \mu G_1(x(t_n), \quad \varphi(t_n), \mu)$$
$$\varphi(t_n) = \pi n + \mu F(x(t_n), \quad \varphi(t_n), \mu)$$
$$z(t_n) = \pi n, \quad x(0) = x_0, \quad M_1 = M_1(z, \alpha)$$

Here x and y are, as before, vector functions of arbitrary finite dimensions, z and φ are scalar phases, X, Y, Z and Φ are functions depending on arguments x, yM_1, φ, zM_1 and μ, Φ, G, F and Z are bounded functions, and furthermore $F(x, \varphi, \mu)$ has bounded partial derivatives with respect to its first two arguments; the requirements on the functions X, Y and G are the same as in system (1.1).

The function $M_1(z, \alpha)$ is shown on Fig.2: the switchover point α may depend on slow variables.

We introduce a new phase $\varphi_1 = \varphi - \mu F$ and a slow variable $\theta = z - \varphi_1$. We then change to the phase

$$\varphi_2 = \begin{cases} \pi n + \dfrac{\pi(\varphi_1 - \pi n)}{2(\alpha - \theta)}, & \pi n \leqslant \varphi_1 \leqslant \pi n + \alpha - \theta \\ \pi n + \dfrac{\pi}{2}\left(1 + \dfrac{\varphi_1 - \pi n - \alpha + \theta}{\pi - \alpha + \theta}\right), & \pi n + \alpha - \theta \leqslant \varphi_1 \leqslant \pi(n+1) \end{cases}$$

to which there corresponds a discontinuous frequency

$$U = \begin{cases} \dfrac{\pi}{2(\alpha - \theta)}, & \pi n \leqslant \varphi_2 \leqslant \pi(n + {}^1/_2) \\ \dfrac{\pi}{2(\pi - \alpha + \theta)}, & \pi(n + {}^1/_2) \leqslant \varphi_2 \leqslant \pi(n+1) \end{cases}$$

and considering φ_2 to be an independent variable, we obtain the system

$$\frac{dx}{d\varphi_2} = \mu\frac{X}{U} + \mu^2 O(1), \quad \frac{dy}{d\varphi_2} M(\varphi_2) = \mu\left\{\frac{Y}{U} + \mu O(1)\right\} M(\varphi_2) \tag{2.2}$$
$$\frac{d\theta}{d\varphi_2} M(\varphi_2) = \mu\left\{\frac{Z - \Phi}{U} + \mu O(1)\right\} M(\varphi_2)$$
$$y(\pi n) = G(x(\pi n)) + \mu O(1), \quad \theta(\pi n) = 0; \quad x(0) = x_0$$

to which one can apply the theorem that has been proved.

3. Equations of the form (2.1) appear naturally in considerations of oscillations of vibrocollisonal systems with vanishing velocity restitution coefficients during a collision, if one restricts oneself to periodic motion regimes with contact zones and uses the approach of /3, 4/ associated with non-analytic changes of variables. As an example we consider the simplest vibrocollisional system with a two-sided restriction and kinematic excitation, shown in Fig.3 ($d_2 - d_1 = 2l$).

Its equations of motion have the form

$$s^{\cdot\cdot} = -k^2(1-\mu)s + k^2\mu(1-\mu)r + k^2 a(1-\mu)\sin\omega t \tag{3.1}$$
$$r^{\cdot\cdot} = \begin{cases} k^2 s - k^2\mu r - k^2 a \sin\omega t, & |r| < l \\ 0, & |r| = l \end{cases}$$

$$r = \pm l, \quad s = \mu r + a\sin\omega t, \quad s^{\cdot} = 0 \tag{3.2}$$

Here s is the coordinate of the centre of mass of the system, r is the relative coordinate of the body of mass M_0 in the gap, $k = \sqrt{c/M_0}$ is the frequency of free oscillations of the body of mass $M_0, \mu = m/(M_0 + m)$ and a/l are small parameters of single order. Conditions (3.2) are separation conditions (the transition from the stage of joint motion to separate motion).

We now change to new phase variables with the aid of the following non-analytical transformation, which for a regime with contact zones and two collisions in one oscillation period of the yoke ensures exact satisfaction of the collision conditions:

$$s = A \sin\psi,\ s^{\cdot} = Ak\cos\psi,\ \Omega = \omega t$$
$$r = l\,(\mathrm{sign}\,(\sin\varphi) - M_2) + \{-l + B\,(\varphi - \pi\,[\varphi/\pi]) - A\sin\psi M_2\}\, M_2$$
$$r^{\cdot} = k\,\{B - A\cos\psi M_2\}\, M_2,\ M_2 = M_2\,(\varphi, \alpha) = M_1\,(\varphi, \alpha)\mathrm{sign}\,(\sin\varphi)$$

Here $[z]$ is the integer part of z and the auxiliary slow variable α is defined by the transcendental equation

$$B\alpha - A\sin\psi = 2l \text{ when } \varphi = \alpha$$

We will consider the most interesting resonant case $\omega - k = k\Delta$, where Δ is a small parameter of order μ. Introducing the slow variable $\sigma = \Omega - \psi$ and the dimenionless time $\tau = kt$, and restricting ourselves to terms of order μ, we obtain a system in the form (2.1):

$$A^{\cdot} = {}^1/_2\mu A\sin 2\psi + \mu r\cos\psi + a\sin(\psi + \sigma)\cos\psi$$
$$\sigma^{\cdot} = \Delta + \mu\sin^2\psi + \mu rA^{-1}\sin\psi + aA^{-1}\sin(\psi + \sigma)\sin\psi$$
$$B^{\cdot}M_2\,(\varphi, \alpha) = \mu A\sin\psi\, M_2^2\,(\varphi, \alpha)$$
$$\varphi^{\cdot}M_2\,(\varphi, \alpha) = \{1 - \mu\,(\varphi - \pi\,[\varphi/\pi])AB^{-1}\sin\psi\, M_2\,(\varphi, \alpha)\}\, M_2\,(\varphi, \alpha)$$
$$\psi^{\cdot} = 1 - \mu\sin^2\psi - \mu rA^{-1}\sin\psi - aA^{-1}\sin(\psi + \sigma)\sin\psi$$

with separation conditions

$$\varphi = \pi n,\ B = A + \mu^2 O\,(1),\ \psi = \pi n + \mu O\,(1)$$

Averaging over the fast phase ψ with the help of the procedures derived above and denoting differentiation with respect to ψ by a prime, we arrive at the equations of the first approximation

$$A_1' = \mu\pi^{-1}\{-2l\sin\alpha_1 - A_1(\alpha_1\sin\alpha_1 + \cos\alpha_1 - 1) - {}^1/_2 A_1\sin^2\alpha_1\} + {}^1/_2 a\sin\sigma_1,\ \sigma_1' = \Delta + {}^1/_2\mu + \mu/\pi\,\{2l/A_1\cos\alpha_1 - \alpha_1\cos\alpha_1 + \sin\alpha_1 - {}^1/_2\alpha_1 + {}^1/_4\sin 2\alpha_1\} + {}^1/_2 aA_1\cos\sigma_1,\ \alpha_1 - \sin\alpha_1 = 2lA_1^{-1} \tag{3.3}$$

Eqs.(3.3) are true for times of the order of $1/\mu$ with errors of the order of μ. Setting the right-hand sides of (3.3) equal to zero, we obtain a system of transcendental equations governing the stationary resonant regime.

Fig.3 shows the dependence of the resonant amplitude of the centre of mass A_1 on the wave selection Δ for $\mu = 0.01$ and $a/l = 0.01$. Here it is necessary to take into account the condition for realizing the simplest vibrocollisional regime with contact zone that we are considering: $0 < \alpha_1 < \pi$; the existence of the given resonsant regime is ensured when the inequalities

$$\mu < \pi^2 a/(8l),\ |\Delta| < \pi a/(4l)$$

are satisfied.

A similar approach can also be used when considering more-complex problems which taking into account the non-linearity of the oscillation exciter.

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